Artificial Intelligence
6 – Online MDP Solving
(2048 and Expectimax)

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INSA 4IR

Markov Decision Processes

States and actions:

- S: set of states
- $Actions(s)$: set of actions available in state s
  - a state with no available actions is a **terminal** state

Transition model:

- $P(s'|s,a)$: probability of reaching state s' after taking action a in state s
  - **markovian**: only depends on current state and action (and not on history)

Reward model:[1]

- $R(s,a,s')$: reward received after taking action a in state s and reaching state s'
  - 1 for reaching the goal, -1 for reaching the negative goal
  - -0.04 for each other transition (action cost, encouraging to reach the goal quickly)

[1] a negative reward can be interpreted as a cost

2048 as an MDP

- **States**: all possible board configurations
- **actions**: subset of {up, down, left, right} (the ones that change the board)
- **Transition model**:
  - first move all tiles in the direction of the action (merging identical tiles)
  - the uniformely select an empty time
  - place a 2 or 4 tile with respective probabilities 0.9 and 0.1
- **reward**: the value of any newly merged tile

Section 1

State Space

Game Dynamics

At each turn the sum of all tiles is incremented by 2 or 4.

- merging two tiles of value x and y gives a tile of value $x + y$ (no changes)
- a new tile is added with value 2 or 4

How much space do I need to produce a single tile?

Assume I just got the SUM 2048 (2^11)

What's the representation of those sums?

- $2048 = 2^{11} =$ 0b100000000000
  - $2048 = 2^{10} + 2^{10}$
  - $2048 = 2^{10} + 2^9 + 2^9$

Minimum number of tiles

Just before having the SUM 2048, I had either:

- the SUM 2046 $(2^{11} - 2)$
- the SUM 2044 $(2^{11} - 4)$

What the most compact representation of those sums?

- $2046 = 2^{11} - 2 =$ 0b011111111110
- $2044 = 2^{11} - 4 =$ 0b011111111100

Minimum number of tiles

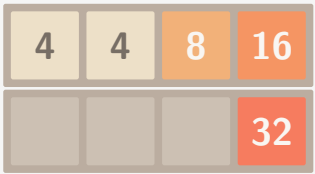
Theorem

To produce the tile 2^N, I need at least the space for $N-1$ tiles.

- $N-2$ *to represent the sum* 2^N-4
- $+1$ *free space to where a new tile (4) will be added*

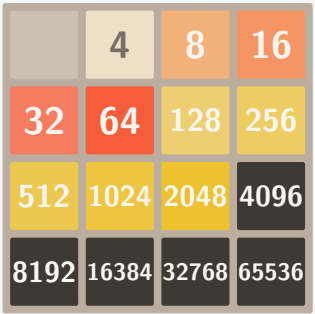
32 (2^5) requires 4 a board of size at least 4

- $32 = 16+8+4+4$
- $2^5 - 4 = 2^4 + 2^3$



Biggest tile

The board has 16 spaces, so we can in the best case produce a tile of value 2^{17} (131072).



In situation (SUM $= 2^{17} - 4$), we can either

- get a new tile of value 2
  - GAME OVER
- get a new tile of value 4
  - we will produce the tile 2^{17}

Even in the best situation above, we have 90% chance of losing the game before getting the tile 2^{17}.

Winning Conditions

What are the expectations of the players?

- humans (experienced players):
  - 2048 (common)
  - 4096 (challenging)
  - 8192 (hard)
- machines:
  - 8192 (common)
  - 16384 (challenging)
  - 32768 (hard)
  - 65536 (never seen personally[2])

[2] A very involved algorithm, with heavy offline computation for end-games, claims to reach 65536 with a probability of 5.8%

Game length

How many turns are necessary to reach the SUM 32768?

- each turn we add a tile of value 2 or 4
  - the average increment is $2.2 = 2 \times 0.9 + 4 \times 0.1$
- Reaching the SUM 32768 requires on average: 14894.5 turns (32768 / 2.2)

State space size

How many possible states are there in the game?

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Having N distinct tiles on the board, how many possible states are there?

- $N = 1$: 16 possible states
- $N = 2$: 16 x 15 possible states
- $N = 3$: 16 x 15 x 14 possible states

State space size

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Having N distinct tiles on the board, how many possible states are there?

- $N = 1$: 16 possible states
- $N = 2$: 16 x 15 possible states
- $N = 3$: 16 x 15 x 14 possible states

$$\frac{16!}{(16-N)!}$$

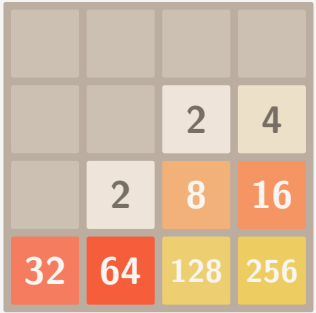
- $N = 8$: 5×10^8
- $N = 12$: 9×10^{12}

Clearly, we can't store all possible states in memory.

Section 2

State representation

State representation



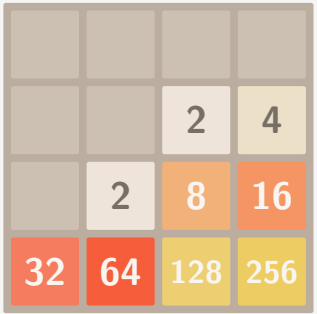
How to encode this in a computer?

- 4 by 4 matrix
- of integers

What's the number of bits necessary to store the integers?

- 2048 -> 11 bits (two bytes)
- 4096 -> 12 bits (two bytes)
- 8192 -> 13 bits (two bytes)
- 16384 -> 14 bits (two bytes)
- 32768 -> 15 bits (two bytes)
- 65536 -> 16 bits (two bytes)
- 131072 -> 17 bits (three bytes)

State representation

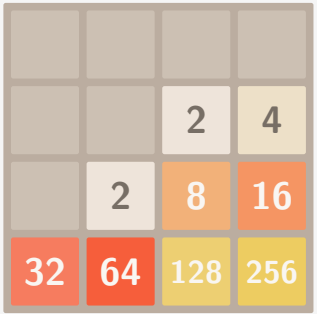


If we settle for 16 bits, we can represent up to 65536.
Total memory for the board:

- 16 bits x 16 tiles = 256 bits = 32 bytes
- 4 machine words (64 bits)

Could we do better?

State representation



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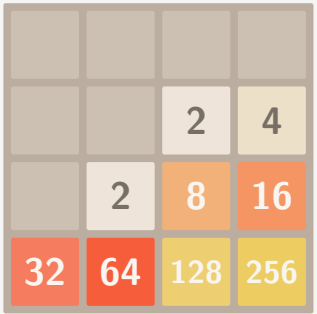
Could we do better?

- tiles are powers of 2
  - $16 = 2^4$
  - $2048 = 2^{11}$
  - $4096 = 2^{12}$
  - $65536 = 2^{16}$
- What if we stored the exponent?
  - the number x in the matrix represents the tile 2^x
  - only 4 bits necessary to store the exponent
- Memory for the board:
  - 4 bits x 16 tiles = 64 bits = 8 bytes

State representation



Yet, 4 bits are annoying to manipulate.

- a CPU is only allowed to manipulate (load/store) bytes
- this course is not about bit twiddling

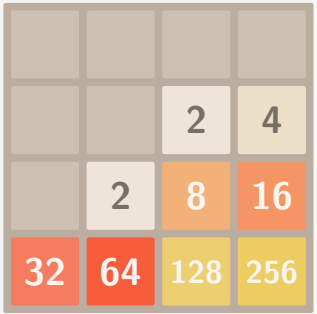
Lets use:

- one byte for each tile
- with the exponent trick

Board:

- 4 x 4 matrix
- each with one 8-bits unsigned integer (u8 in Rust)
- the number x in the matrix represents the tile 2^x
- total size of the board: 16 bytes

State representation



Encoding of the board:

```
[[0,0,0,0],
 [0,0,1,2],
 [0,1,3,4],
 [5,6,7,8]]
```

Section 3

Utility estimation

Performance measure

Your score is the sum of all tiles that you have produced over the entire game.

Could we estimate the score at a given state?

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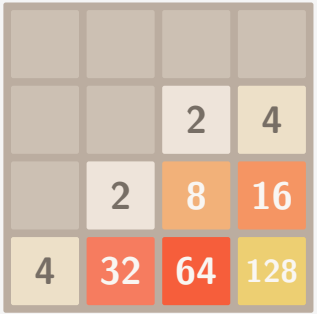

- score(2) = 0 (just popped)
- score(4) = 4 (assuming we merged 2 and 2)[a]
- score(8) = 8 + 2 x score(4) = 16
- score(16) = 16 + 2 x score(8) = 48
- score(32) = 32 + 2 x score(16) = = 128
- score(64) = 64 + 2 x score(32) = 320
- score(128) = 128 + 2 x score(64) = 768
- ...

[a] the real average score for a 4 tile is 4×0.9 as it would just have appear 10% of the time

Score(board) = 4 + 4 + 48 + 128 + 320 + 768 = 1272

Performance measure

Observations:

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4. the SUM increases linearly with the number of turns

$\Rightarrow$ the number of turns is an almost perfect proxy for the score

- our performance measure: the number of turns played

Solved?

$$\mathit{perf}(b) = \frac{\mathit{SUM}(b)}{2.2}$$

$$U(b) = \begin{cases} \mathit{perf}(b) & \text{if } terminal(b) \\ \max_a \sum_{b'} P(b'|b,a) \times U(b') & otherwise \end{cases}$$

Solved?

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Intractable, requires exploring all possible state sequences until a terminal state

Utility estimation vs Performance measure

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- NOTHING!
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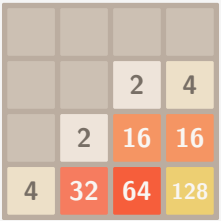
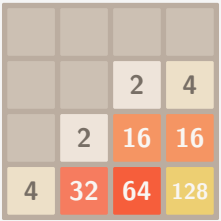
Utility function: combination of numerical values **correlating** with the *life expectancy*

Utility estimation



Some simple metrics:

Utility estimation



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- number of empty tiles
- number adjacent tiles with the same value

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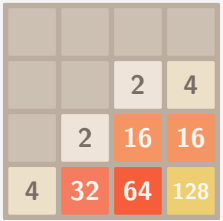

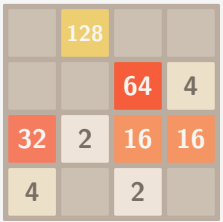

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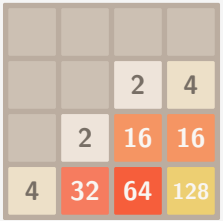

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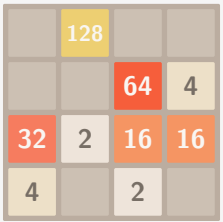


Need to account for:

- the structure of the board
- emphasizing large tiles

Monotonicity (naive counting)

monotonicity: count the number of increases and decreases in the sequences

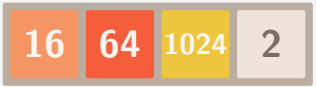
- increases: 3
- decreases: 0
- monotonicity penalty: 0 (min)

- increases: 2
- decreases: 1
- monotonicity penalty: 1 (min)

Differential Monotonicity

Weighting: to emphasize the importance of large tiles[3]

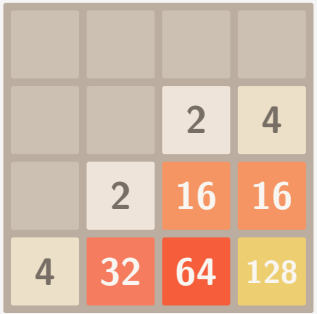
- increases: $(4-2)+(16-4)$
- decreases: $(16-8)$
- monotonicity penalty: 8 (min)



- increases: $(64-16)+(1024-64)$
- decreases: $(1024-2)$
- monotonicity penalty: 1022 (min)

[3] Slide presents the main idea but implementations may more complex

Monotonocity (board)



Monotocity measure for the board:

- sum of the monotonicity measure for each row and column
- favors placing large tiles in a corner

Utility measure

$$\begin{aligned}\hat{U}(b) = w_1 &\times \textit{empty-tiles}(b) \\ + w_2 &\times \textit{monotonicity}(b) \\ + w_3 &\times \textit{adjacent-tiles}(b)\end{aligned}$$

How to set the weights w_1, w_2, and w_3?

- trial and error (by hand)
- trial and error (optimization algorithm)[4]
- machine learning (reinforcement learning)

[4] Someone did it for us https://github.com/nneonneo/2048-ai

Section 4

Algorithms

Structure of a turn

A turn is composed of:

- the application of an action a, selected by the agent
  - $s' \leftarrow result(s, a)$
- the addition of a random time
  - $s'' \leftarrow add\_tile(s')$ with probability $P(s''|s')$

Greedy

- **Greedy algorithm**: at each turn, choose the action that maximizes the approximated utility of the resulting state

$$\textit{greedy}(s) = \arg\max_a \hat{U}(result(s, a))$$

Complexity: $O(1)$ (at most 4 evaluations of the utility function)

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Surely we can do better than that!

From Bellman to Expectimax

Recall the Bellman equation:

$$U(s) = \max_{a \in Actions(s)} \sum_{s'} P(s'|s,a) \left[R(s,a,s') + \gamma U(s') \right]$$

For 2048, we can simplify it (no rewards nor discounts)

$$U(s) = \max_{a \in Actions(s)} \sum_{s'} P(s'|s,a) U(s')$$

From Bellman to Expectimax

Expectimax: approximates the gathering the average utility of the states reachable with a fixed number of actions.

Utility estimated by EXPECTIMAX, with lookahead (depth) d

$$U_{EM}(s,d) = \begin{cases} \hat{U}(s) & \text{if } d = 0 \\ \max_a \sum_{s'} P(s'|s,a) U_{EM}(s', d-1) & otherwise \end{cases}$$

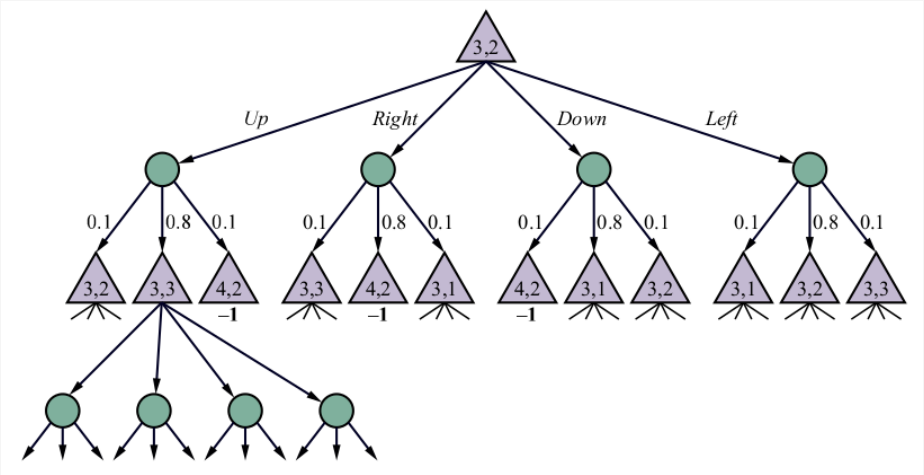
Expectimax algorithm

$$expectimax(s, d) = \arg\max_a eval\_chance(result(a), d-1)$$

$$eval\_chance(s) = \begin{cases} \hat{U}(s) & \text{if } d = 0 \\ \sum_{s'} P(s'|s) \times eval\_max(s', d-1) & otherwise \end{cases}$$

$$eval\_max(s, d) = \max_a eval\_chance(result(s, a), d-1)$$

Expectimax: tree



Expectimax: complexity

- Time complexity (size of the explored tree): $O(b^d)$
  - b: branching factor (number of actions x number of possible new tiles)
  - d: depth of the tree (number of turns)
- Space complexity: $O(d)$ (depth of the tree, like DFS)

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Typical values for 2048:

- 3 available actions
- 6 empty tiles (less in late game)
  - 12 possible new tiles (2 or 4)
- $b = 3 \times 12 = 36$ (a lot of variations across the game)

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Explored states:

- $d = 1$: $36^1 = 36$
- $d = 2$: $36^2 = 1296$
- $d = 3$: $36^3 = 46656$
- $d = 4$: $36^4 = 1679616$
- $d = 5$: $36^5 = 60466176$

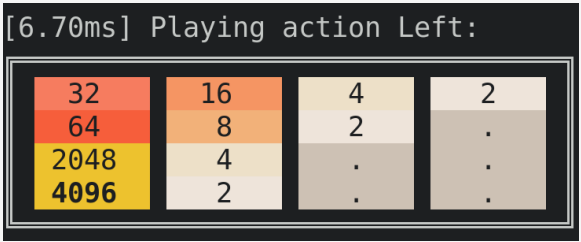
Next week: 2048 lab

Given:

- game logic
- utility function implementation
- algorithm evaluation code

To do:

- implement the greedy algorithm
- implement the Expectimax algorithm
- evaluate the performance of the algorithm
- improve and compete?



Going further

- **transposition table**: caching the results of the eval_max/eval_chance functions
  - exploit redundancy in the search tree
- **iterative deepening**: gradually increase the depth of the search
  - deeper search on states with lower branching factor (typical in end games)
- **approximation**: early termination the search
  - on very unlikely states (e.g. we got 6 tiles of value 4 in a row)
  - on states with a very low utility (e.g. recovery is unlikely and this branch will never be selected)
- **heuristic**: tuning the heuristic utility function
  - offline by hand or optimization (has already been done for you)
  - online with reinforcement learning (out of scope for this course)
    - The heuristic is, e.g., a neural network that is trained while playing the game[5]

[5] https://arxiv.org/abs/1604.05085